Noncommutative differential calculus.

Definition. Let A and M be $\mathbb{Z}/2\mathbb{Z}$-graded algebra and a B-bimodule, respectively. $D : A \to M$ is called an (odd) derivation if for any homogeneous element b,

$$D(a_1 a_2) = D(a_1)a_2 + (-1)^{\bar{a}_1} a_1 D(a_2).$$

Exercise 45. Show that for any odd operator F on a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V

$$D(a) := F \circ a - (-1)^{\bar{a}} a \circ F$$

defines an odd derivation $D : \mathrm{End}(M) \to \mathrm{End}(M)$, and if $F^2 = 1$, $D^2 = 0$.

Exercise 46. Show that for every algebra A there is a universal derivation $d : A \longrightarrow \Omega^1 A$, i.e.

$$\begin{tikzcd}
 & \Omega^1 A \arrow[dr, dashed, "\exists!\ A\text{-bimodule map } \varphi_D"] & \\
A \arrow[ur, "d"] \arrow[rr, "\forall D"'] & & M \\
 & & \text{derivation}
\end{tikzcd}$$

Solution. $\Omega^1 A = \ker(A \otimes A \xrightarrow{m} A)$

$$d : A \longrightarrow \Omega^1 A, \quad a \longmapsto 1 \otimes a - a \otimes 1.$$

$$\Omega^1 A \ni \sum_i a_i' \otimes a_i'' = \sum_i a_i' \otimes a_i'' - \sum_i a_i' a_i'' \otimes 1 = \sum_i a_i' \, d a_i''$$

$$\varphi \in \mathbf{Bimod}_A(\Omega^1 A, M) = \mathbf{Der}(A, M) \ni D$$

$$\varphi_D\Big(\sum_i a_i' \otimes a_i''\Big) := \sum_i a_i' D(a_i''), \qquad D_\varphi(a) := \varphi(1 \otimes a).$$

$$\varphi_{D_\varphi}\Big(\sum_i a_i' \otimes a_i''\Big) = \sum_i a_i' D_\varphi(a_i'') = \sum_i a_i' \varphi(1 \otimes a_i'')$$

$$= \varphi\Big(\sum_i a_i' \otimes a_i''\Big).$$

$$D_{\varphi_D}(a) = \varphi_D(1 \otimes a) = 1 \cdot D(a) = D(a). \quad \square$$

Exercise 47. Show that $\Omega^1 A / [A, \Omega^1 A]$ for a commutative algebra A admits a universal derivation on A with values in symmetric A-bimodules.

Solution. $\Omega^1 A/[A,\Omega^1 A]$ as a symmetric A-bimodule admits a universal derivation induced by φ:

$\omega = \sum_i a_i' \otimes a_i'' \in \Omega^1 A \Rightarrow [a, \omega] = \sum_i a a_i' \otimes a_i'' - \sum_i a_i' \otimes a_i'' a$

$= \sum_i a_i' a \otimes a_i'' - \sum_i a_i' \otimes a a_i''$

$\varphi_D([a, \omega]) = \sum_i a_i' a\, D(a_i'') - \sum_i a_i' D(a_i'' a) = \underbrace{\sum_i a_i' a_i''}_{= 0} D(a) = 0.$ $\blacksquare$

Exercise 48. Show that for A commutative $\Omega^1 A \subset A \otimes A$ is an ideal and $(\Omega^1 A)^2 = [A, \Omega^1 A]$.

Solution. $da\, da' = (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1)$

$= 1 \otimes aa' - a' \otimes a - a \otimes a' + aa' \otimes 1$

$= (aa' \otimes 1 - a' \otimes a) - (a \otimes a' - 1 \otimes a'a)$

$= [a, a' \otimes 1] - [a, 1 \otimes a'] = [a, a' \otimes 1 - 1 \otimes a']$

$= [a, -da'] \in [A, \Omega^1 A] \Rightarrow (\Omega^1 A)^2 = [A, \Omega^1 A]$. $\blacksquare$

Corollary. $\Omega^1 A / [A, \Omega^1 A] \cong \Omega^1 A / (\Omega^1 A)^2 = I_\Delta / I_\Delta^2$ (conormal module).

Definition. A differential graded algebra $(\Omega^{\bullet}, d)$ is a $\mathbb{Z}$-graded vector space $\Omega^{\bullet} = \bigoplus_{n\in\mathbb{Z}} \Omega^n$ s.t. $\Omega^p \Omega^q \subset \Omega^{p+q}$ and $d\Omega^p \subset \Omega^{p+1}$ is an odd derivation s.t. $d^2 = 0$

Exercise 49. Show that for every algebra A there exists a universal dga $\Omega^{\bullet}A$ admitting $d: A \to \Omega^{\bullet}A$.
(dg-envelope of A)

Solution. $\Omega^{\bullet}A := \left\langle a, da \;\middle|\; \begin{array}{l} d(ca) = c\,da \\ d(a'+a'') = da' + da'' \\ d(a'a'') = da'a'' + a'da'' \end{array} \right\rangle$

$d(a) = da$, $d(da) = 0$.

$\Omega^n A \overset{?}{:=} \langle a_0 da_1 \dots da_n \rangle$

$$a'(a_0 da_1 \dots da_n) = a'a_0 da_1 \dots da_n,$$

$$(a_0 da_1 \dots da_n) a' = a_0 da_1 \dots da_n a'$$

$$= a_0 da_1 \dots da_{n-1} d(a_n a') - a_0 da_1 \dots da_{n-1} a' \cdot da_n$$

$$\dots = (-1)^n a_0 a_1 da_2 \dots da_n da'$$

$$+ \sum_{i=1}^{n-1} (-1)^{n-i} a_0 da_1 \dots d(a_i a_{i+1}) \dots da_n da'$$

$$+ a_0 da_1 \dots da_{n-1} d(a_n a').$$

$$\Rightarrow (a_0 da_1 \dots da_n)(a_0' da_1' \dots da_m') := (a_0 da_1 \dots da_n \cdot a_0') da_1' \dots da_m'$$

well defined.

$$d(a_0\, da_1 \dots da_n) := da_0\, da_1 \dots da_n \qquad (\Rightarrow d^2 = 0)$$

well defined odd derivation if it kills relations

$$d(d(ca) - c\, da) = d(1 \cdot d(ca) - c\, da) = d(1)\, d(ca) - d(c)\, da = 0$$

$$d(d(a+a') - da - da') = d(1 \cdot d(a+a') - 1 \cdot da - 1 \cdot da') = 0$$

$$d(d(aa') - da \cdot a' - a\, da') = dd(aa') - dd(a) \cdot a' + da \cdot da'$$
$$- da \cdot da' - a\, dda' = 0 .$$

$d : A \to \Omega^1 A$, $a \mapsto da$.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\forall \varphi"'] & \Omega^\bullet A \arrow[d, dashed, "\exists!\ \text{dga map}"] \\ & \Omega^\bullet \end{tikzcd}$$

algebra map

$a \mapsto \varphi(a)$

$$\begin{array}{cc} a & da \\ \downarrow & \downarrow \\ \varphi(a) & d\varphi(a) \end{array}$$

well defined. $\square$

Exercise 51. Given $D \in \mathrm{Der}(A,A)$ show that the bimodule map $\iota_D : \Omega^1 A \to A$ extends to an odd derivation of $\Omega^\bullet A$ of degree -1 (denoted also by ι_D). Check the $\mathcal{L}_D := \iota_D \circ d + d \circ \iota_D$ is an even derivation of degree zero and

$$[\iota_D, \iota_{D'}] = 0, \quad [\mathcal{L}_D, \iota_{D'}] = \iota_{[D,D']}, \quad [\mathcal{L}_D, \mathcal{L}_{D'}] = \mathcal{L}_{[D,D']}.$$

Remark. If $A = C(X)$, X a compact we can identify $A^{\otimes n+1}_{\max}$ with $C(X \times \dots \times X)$ (by the Stone-Weierstrass). Then $\Omega^n A$ can be identified with the space of functions on X^{n+1} vanishing on all diagonals.

$$(fg)(x_0, \dots, x_p, x_{p+1}, \dots, x_{p+q})$$

$$= f(x_0, \dots, x_p)\, g(x_p, \dots, x_{p+q})$$

$$(df)(x_0, \dots, x_n) = \sum_{k=0}^{n} (-1)^k f(x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$$

$d^2 = 0$ obvious.

This complex is acyclic (cohomology $\cong$ ($\mathbb{C}$ in degree 0)).

To get something non-acyclic one has to quotient by functions vanishing when restricted to a cartesian power of some element of some open covering of X. The cohomology of the quotient = Alexander-Spanier cohomology of X.

Problem. This makes no sense for a noncommutative algebra A.

Remark. For $A = C^\infty(X)$, X a compact manifold then again $A^{\otimes n+1}$ (Fréchet tensor product) can be identified with $C^\infty(X \times \dots \times X)$. One can repeat the same as above replacing opens containing $(x_0, \dots, x_n)$ by "infinitesimal simplices". The quotient becomes the De Rham complex.

Problem. This also doesn't make sense for noncommutative algebras.

But try to mimic integration of differential forms in such a way, that it makes sense for the dga-envelope.

Definition. An <u>n-dimensional cycle</u> is a complex dga $(\Omega^\bullet, d)$ equipped with an <u>integral</u> $\int$, i.e. a linear map $\int : \Omega^\bullet \longrightarrow \mathbb{C}$ s.t. $\int \omega_k = 0$ for $k < n$, and for homogeneous elements

$$\int \omega_k \omega_\ell = (-1)^{k\ell} \int \omega_\ell \omega_k \quad \text{and} \quad \int d\omega_{n-1} = 0 .$$

It is denoted $(\Omega^\bullet, d, \int)$.

Example. $\Omega^\bullet = \Omega^\bullet_{dR}(X) \otimes_{\mathbb{C}} M_n(\mathbb{C}) = M_n(\Omega^\bullet_{dR}(X))$.

for a closed oriented smooth manifold X of dim $= n$,

$$\int \omega_k := \begin{cases} \int_X tr(\omega_k) & k = n, \\ 0 & k \neq n . \end{cases}$$

Symmetry comes from skew symmetry of the exterior product and symmetry of the trace of matrices. Closedness comes from the Stokes thm

Cycles from Fredholm modules.

Definition. Let A be an algebra. An odd (even) Fredholm module over A is an operator (odd operator) F on a separable Hilbert space H ($\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $H = H^0 \oplus H^1$), s.t. $F = F^*$, $F^2 = 1$ and a representation (even representation) $\varphi : A \to B(H)$ s.t.

$$[F, \varphi(A)] \subset \mathcal{K}(H).$$

Often Fredholm modules are constructed from pre-Fredholm modules, where one asks only that $\varphi(A)(F-F^*)$, $\varphi(A)(F^2-1) \subset K(H)$ for some Fredholm operator. Then

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \quad \text{on} \quad H = H^0 \oplus H^1$$

where $P \in \mathbf{Fred}(H^0, H^1)$, $Q \in \mathbf{Fred}(H^1, H^0)$ and $1-PQ$, $1-QP$ are compact.

Doubling. $\widetilde{H}^0 := H^0 \oplus H^1$, $\widetilde{H}^1 := H^1 \oplus H^0$

$$\widetilde{\varphi}_0(a) := \begin{pmatrix} \varphi_0(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{\varphi}_1(a) := \begin{pmatrix} \varphi_1(a) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\widetilde{F} := \begin{pmatrix} 0 & \widetilde{Q} \\ \widetilde{P} & 0 \end{pmatrix}, \text{ where}$$

$$\widetilde{P} := \begin{pmatrix} P & 1-PQ \\ 1-QP & QPQ-2Q \end{pmatrix}, \quad \widetilde{Q} := \begin{pmatrix} 2Q-QPQ & 1-QP \\ 1-PQ & -P \end{pmatrix}$$

Exercise 52. Check that $\widetilde{F}^* = \widetilde{F}$, $\widetilde{F}^2 = 1$, and $[\widetilde{F}, \widetilde{\varphi}(A)] \subset K(\widetilde{H})$.

Example.

[Atiyah] $A = C^\infty(X)$, X Riemannian compact manifold, H^0, H^1 L^2-sections of Hermitian vector bundles over X, P elliptic pseudodifferential operator of order 0 $P \in \mathrm{Fred}(H^0, H^1)$, $F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$, Q a parametrix of P (i.e. $1-PQ$, $1-QP$ compact).

Remark. In general, we think of Fredholm modules as "generalized" or "noncommutative" elliptic operators.

Construction of the cycle.

Assume $F^2 = 1$, $T \in B(H)$. Then

The condition $[F, \varphi(A)] \subset K(H)$ has to be strengthen to $[F, \varphi(A)] \subset \mathcal{L}_{n+1}(H)$ (i.e. $\mathcal{L}_{n+1}(H)^{n+1} \subset T(H)$)

we assume that n is odd for odd Fredholm modules.

$$\Omega^n := \{ \varphi(a_0)[F,\varphi(a_1)] \dots [F,\varphi(a_n)] \in B(H) \}$$

$$\Omega^0 = \varphi(A) \subset B(H)$$

$$d(\varphi(a)) := i\,[F,\varphi(a)] \qquad \left(\begin{array}{l} i \text{ makes it compatible with } * \\ \text{i.e. } d(a^*) = d(a)^* \end{array} \right)$$

$$d\omega = i[F,\omega] \,, \quad \omega_p \in \Omega^p \,, \quad \omega_q \in \Omega^q \Rightarrow \omega_p \omega_q \in \Omega^{p+q} .$$

$$\begin{cases} \int \omega_n := -\frac{i}{2} \mathrm{Tr}(F\, d\omega_n) & \text{for } n \text{ odd.} \\ \int \omega_k := 0 \quad \text{for } k \neq n & \end{cases}$$

$$\begin{cases} \int \omega_n := -\frac{i}{2} \mathrm{Tr}(\gamma F\, d\omega_n) & \text{for } n \text{ even} \\ \int \omega_k := 0 \quad \text{for } k \neq n & \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ on } H^0 \oplus H^1 . \end{cases}$$

Closedness obvious. Graded trace property?

$\underline{k+l=n \text{ odd}}$, k odd, l even,

$$\int \omega_k \omega_l = -\frac{i}{2} \mathrm{Tr}\left(F\, d(\omega_k \omega_l)\right) = -\frac{i}{2} \mathrm{Tr}\left(F\, d\omega_k \cdot \omega_l - F\omega_k \cdot d\omega_l\right)$$

$$= -\frac{i}{2} \mathrm{Tr}\left(-d\omega_l F \omega_k + \omega_l F\, d\omega_k\right) = \dots$$

since $\frac{k}{n+1} + \frac{l+1}{n+1} = 1$ and $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$ for $S \in L_p(H)$, $T \in L_q(H)$.

because $[F, \varphi(a)] \in L_{n+1}(H)$

where $\frac{1}{p} + \frac{1}{q} = 1$

$$\dots = -\frac{i}{2} \mathrm{Tr}\left(F\, d\omega_l \cdot \omega_k + F\omega_l\, d\omega_k\right) = -\frac{i}{2} \mathrm{Tr}\left(F\, d(\omega_l \omega_k)\right) = \int \omega_l \omega_k$$

$\underline{k+l = n \text{ even}}$ similar with use $\chi\omega_k = (-1)^k \omega_k \chi$.

Theorem. [Connes–Langsmann] For any $a_0, \dots, a_n \in M_m(C^\infty(\mathbb{T}^n))$,
$H = L_2(\mathbb{T})^m$

$$\int \varphi(a_0)\, d\varphi(a_1) \dots d\varphi(a_n) = (2i)^{\lfloor n/2 \rfloor} \frac{\mathrm{vol}(S^{n-1})}{n(2\pi)^n} \int_{\mathbb{T}^n} \mathrm{tr}(a_0\, da_1 \wedge \dots \wedge da_n).$$

Idea of the proof: Fourier series.

Definition.

Let $p \in M_m(A)$ be a projection. For $A = C^\infty(X)$, X a compact oriented manifold we define

$$ch(p) := \sum_{k=0}^{\infty} \frac{1}{k!} \, tr\left(p (dp)^{2k}\right) \in \Omega^{\bullet}_{dR}(X)$$

Exercise 53. Compute the curvature of the Levi-Civita connection on pA^m

$$\nabla(ps) = p\, d(ps)$$

Solution. $\nabla^2(ps) = pd(p\cdot p(d(ps)))$

$= pd(p(d(ps))) = pdp(d(ps)) = pdp(d(p\cdot ps))$

$= p\cdot dp\cdot dp(ps) + p\cdot dp\cdot ps$

$pdp\,p = p(d(p\cdot p) - p\,dp) = p\,d(p\cdot p) - (p\cdot p)dp$

$= pdp - pdp = 0$

$\Rightarrow \nabla^2(ps) = p(dp)^2\cdot(ps). \Rightarrow R_\nabla = p(dp)^2.$ $\blacksquare$

Exercise 54. Show that in $\Omega^{\cdot}_{dR}(X)$

$$d\, ch(p) = 0.$$

Solution. $d\left(tr\left(p(dp)^{2k}\right)\right) = tr\left(dp\,(dp)^{2k}\right)$

$$= tr\left(dp^{2k+1}\right) = tr\left((p+(1-p))\,dp^{2k+1}\right)$$

$$= tr\left(p\,dp^{2k+1}\right) + tr\left((1-p)\,dp^{2k+1}\right)$$

How to commute p and dp ?

$$p\cdot p = p \Rightarrow dp\cdot p + p\cdot dp = dp \Rightarrow p\cdot dp = dp\cdot(1-p)$$

$$(1-p)\,dp = dp\cdot p$$

$\Longrightarrow$

$$\operatorname{tr}(p\,dp^{2k+1}) = \operatorname{tr}(p\cdot p(dp)^{2k+1}) = \operatorname{tr}(p(dp)^{2k+1}p)$$

$$= \operatorname{tr}(p(1-p)(dp)^{2k+1}) = 0$$

$$\operatorname{tr}((1-p)dp^{2k+1}) = \operatorname{tr}((1-p)\cdot(1-p)dp^{2k+1}) = \operatorname{tr}((1-p)dp^{2k+1}p) = 0$$

$$\Rightarrow \quad d(p(dp)^{2k}) = 0 \quad \Rightarrow \quad d\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\, p(dp)^{2k}\Big) = 0 . \ \square$$

Exercise 55. Show that for matrix projections p and q

$$\operatorname{ch}(p \oplus q) = \operatorname{ch}(p) + \operatorname{ch}(q), \quad \operatorname{ch}(1_m) = m$$

Solution. $(p\oplus q)(d(p\oplus q))^{2k} = (p\oplus q)(dp \oplus dq)^{2k}$

$$= (p\oplus q)(dp^{2k} \oplus dq^{2k}) = p(dp)^{2k} \oplus q(dq)^{2k}$$

$$\mathrm{ch}(1_m) = \mathrm{tr}\left(\sum_{k=0}^{\infty} 1_m \cdot (d1_m)^{2k}\right) = \mathrm{tr}(1_m) = m. \quad \square$$

Corollary. $\mathrm{ch}(p) = \mathrm{tr}(\exp(R_\nabla))$

$$K_0(C^\infty(X)) \ni [p\, C^\infty(X)^n] \longmapsto [\mathrm{ch}(p)] \in H^{ev}_{dR}(X)$$

is a well defined homomorphism of abelian groups. $\square$

Definition. The above map is called the <u>Chern character</u>.

Corollary. For $X = \mathbb{T}^n$, $H = L_2(\mathbb{T})^m$, n even

$$\langle [\mathrm{ch}(p)], [\mathbb{T}^n] \rangle = \frac{1}{(\frac{n}{2})!} \frac{n \cdot (2\pi)^n \cdot (-1)^{\frac{n}{2}}}{(2i)^{\frac{n}{2}} \mathrm{vol}(S^{n-1})} \int \varphi(p) [F, \varphi(p)]^n .$$

Exercise 56. Let $A = C^\infty(S^2)$ where

$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and

$$p := \frac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix} .$$

check that $p^2 = p = p^*$ and compute

$$\langle [\mathrm{ch}(p)], [S^2] \rangle .$$